Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Let P be arbitrary point P in $\triangle ABC$ with sidelengths a,b,c .

a) Find minimal value of

$$F(P) := \frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)};$$

b) Prove inequality $\frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)} \geq 36r,$ where r is inradius.

Solution.

a) Let F be area of the triangle.

Applying Cauchy Inequality to triples $\left(\frac{a}{\sqrt{d_a}}, \frac{b}{\sqrt{d_b}}, \frac{c}{\sqrt{d_c}}\right), \left(\sqrt{\frac{d_a}{h_a}}, \sqrt{\frac{d_b}{h_b}}, \sqrt{\frac{d_c}{h_c}}\right)$

we obtain

$$F(P) = \left(\frac{a^2}{d_a} + \frac{b^2}{d_b} + \frac{c^2}{d_c}\right)\left(\frac{d_a}{h_a} + \frac{d_b}{h_b} + \frac{d_c}{h_c}\right) \geq \left(\frac{a}{\sqrt{h_a}} + \frac{b}{\sqrt{h_b}} + \frac{c}{\sqrt{h_c}}\right)^2 =$$

$$\left(\frac{a\sqrt{a}}{\sqrt{2F}} + \frac{b\sqrt{b}}{\sqrt{2F}} + \frac{c\sqrt{c}}{\sqrt{2F}}\right)^2 = \frac{\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2}{2F}.$$

Equality condition is $\frac{a}{\sqrt{d_a}} \div \sqrt{\frac{d_a}{h_a}} = \frac{b}{\sqrt{d_b}} \div \sqrt{\frac{d_b}{h_b}} = \frac{c}{\sqrt{d_c}} \div \frac{c}{\sqrt{h_c}} \Leftrightarrow$

$\frac{2F}{d_a} = \frac{2F}{d_b} = \frac{2F}{d_c} \Leftrightarrow d_a = d_b = d_c \Leftrightarrow \quad p_a \div p_b \div p_c = a \div b \div c.$

Thus, $\min\limits_{P \in \Delta} F(P) = F(I)$, where I is incenter of ΔABC.

In inequality form :

(**1**)
$$\boxed{\frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)} \geq \frac{\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2}{2F}}.$$

b) Let s be semiperimeter of the triangle. Since

$$\left(\frac{a\sqrt{a} + b\sqrt{b} + c\sqrt{c}}{3}\right)^{\frac{2}{3}} \geq \frac{a+b+c}{3} = \frac{2s}{3} \Leftrightarrow$$

$$\left(\frac{a\sqrt{a} + b\sqrt{b} + c\sqrt{c}}{3}\right)^2 \geq \frac{8s^3}{27} \Leftrightarrow \left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2 \geq \frac{8s^3}{3}$$ then

$\frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)} \geq \frac{4s^3}{3F} = \frac{4s^2}{3r}$ and since $s^2 \geq 27r^2$ we finally obtain

$$\boxed{\frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)} \geq 36r}.$$

Equality occurs iff ΔABC is equilateral and $P = I = G = O = H.$